

Characterization of Closed Vector Fields in Finsler Geometry

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Dedicated to the memory of Prof. Dr. A. Tamim

Abstract. The π -exterior derivative \bar{d} , which is the Finslerian generalization of the (usual) exterior derivative d of Riemannian geometry, is defined. The notion of a \bar{d} -closed vector field is introduced and investigated. Various characterizations of \bar{d} -closed vector fields are established. Some results concerning \bar{d} -closed vector fields in relation to certain special Finsler spaces are obtained. ¹

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Introduction

In the present work, we adopt the pullback approach to Finsler geometry. In Finsler geometry, there is a canonical linear connection (corresponding to the Levi-Civita connection of Riemannian geometry), due to E. Cartan, which is not a connection on the manifold M but is a connection on $\pi^{-1}(TM)$, the pullback of the tangent bundle TM by $\pi : \mathcal{T}M \longrightarrow M$. The Cartan connection plays a key role in this work.

We define the notion of π -exterior derivative \bar{d} , which is a natural generalization to Finsler geometry of the (usual) exterior derivative d of Riemannian geometry. We then introduce and investigate an important class of π -vector fields on a Finsler manifold, which we refer to as \bar{d} -closed vector fields. Various characterizations of such π -vector fields are established. Some results concerning \bar{d} -closed vector fields in relation to certain special Finsler spaces are obtained. The notion of a π -distribution is also introduced and is related to \bar{d} -closed vector fields. It should finally be noted that our investigation is entirely global or intrinsic (free from local coordinates).

The idea of this work is due to Prof. A. Tamim, whom we miss profoundly.

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1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback formalism of Finsler geometry necessary for this work. For more details refer to [1], [2] and [8]. We make the general assumption that all geometric objects we consider are of class C^∞ .

The following notations will be used throughout the present paper:

M : a real differentiable manifold of finite dimension n and of class C^∞ ,

$\mathfrak{F}(M)$: the \mathbb{R} -algebra of differentiable functions on M ,

$\mathfrak{X}(M)$: the $\mathfrak{F}(M)$ -module of vector fields on M ,

$\pi_M : TM \longrightarrow M$: the tangent bundle of M ,

$\pi : \mathcal{T}M \longrightarrow M$: the subbundle of nonzero vectors tangent to M ,

$V(TM)$: the vertical subbundle of the bundle TM ,

$P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$: the pullback of the tangent bundle TM by π ,

$P^* : \pi^{-1}(T^*M) \longrightarrow \mathcal{T}M$: the pullback of the cotangent bundle T^*M by π ,

$\mathfrak{X}(\pi(M))$: the $\mathfrak{F}(TM)$ -module of differentiable sections of $\pi^{-1}(TM)$,

$\mathfrak{X}^*(\pi(M))$: the $\mathfrak{F}(TM)$ -module of differentiable sections of $\pi^{-1}(T^*M)$,

i_X : interior product with respect to $X \in \mathfrak{X}(M)$,

df : the exterior derivative of $f \in \mathfrak{F}(M)$,

$d_L := [i_L, d]$, i_L being the interior derivative with respect to the vector form L .

Elements of $\mathfrak{X}(\pi(M))$ will be called π -vector fields and will be denoted by barred letters \overline{X} . Tensor fields on $\pi^{-1}(TM)$ will be called π -tensor fields. The fundamental π -vector field is the π -vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$. The lift to $\pi^{-1}(TM)$ of a vector field X on M is the π -vector field \overline{X} defined by $\overline{X}(u) = (u, X(\pi(u)))$. The lift to $\pi^{-1}(TM)$ of a 1-form ω on M is the π -form $\overline{\omega}$ defined by $\overline{\omega}(u) = (u, \omega(\pi(u)))$.

We have the following short exact sequence of vector bundles, relating the tangent bundle $T(\mathcal{T}M)$ and the pullback bundle $\pi^{-1}(TM)$:

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho = (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) = j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \longrightarrow T_u(T_{\pi_M(v)}M)$. The vector 1-form J on TM defined by $J = \gamma \circ \rho$ is called the natural almost tangent structure of TM and the vertical vector field \mathcal{C} on TM defined by $\mathcal{C} := \gamma \circ \overline{\eta}$ is called the fundamental or the canonical (Liouville) vector field [4].

Let ∇ be a linear connection (or simply a connection) in the pullback bundle $\pi^{-1}(TM)$. We associate to ∇ the map

$$K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto \nabla_X \overline{\eta},$$

called the connection (or the deflection) map of ∇ . A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space to M at u . The connection ∇ is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \quad \forall u \in \mathcal{T}M.$$

If M is endowed with a regular connection, then the vector bundle maps

$$\begin{aligned}\gamma : \pi^{-1}(TM) &\longrightarrow V(TM), \\ \rho|_{H(TM)} : H(TM) &\longrightarrow \pi^{-1}(TM), \\ K|_{V(TM)} : V(TM) &\longrightarrow \pi^{-1}(TM)\end{aligned}$$

are vector bundle isomorphisms. Let us denote $\beta = (\rho|_{H(TM)})^{-1}$, then

$$\rho\circ\beta = id_{\pi^{-1}(TM)}, \quad \beta\circ\rho = \begin{cases} id_{H(TM)} & \text{on } H(TM) \\ 0 & \text{on } V(TM) \end{cases} \quad (1.1)$$

The classical torsion tensor \mathbf{T} of a regular connection ∇ is defined by

$$\mathbf{T}(X, Y) = \nabla_X \rho Y - \nabla_Y \rho X - \rho[X, Y] \quad \forall X, Y \in \mathfrak{X}(TM).$$

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors, denoted by Q and T respectively, are defined by

$$Q(\bar{X}, \bar{Y}) = \mathbf{T}(\beta\bar{X}\beta\bar{Y}), \quad T(\bar{X}, \bar{Y}) = \mathbf{T}(\gamma\bar{X}, \beta\bar{Y}) \quad \forall \bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M)).$$

The classical curvature tensor \mathbf{K} of the connection ∇ is defined by

$$\mathbf{K}(X, Y)\rho Z = -\nabla_X \nabla_Y \rho Z + \nabla_Y \nabla_X \rho Z + \nabla_{[X, Y]}\rho Z \quad \forall X, Y, Z \in \mathfrak{X}(TM).$$

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted by R , P and S respectively, are defined by

$$R(\bar{X}, \bar{Y})\bar{Z} = \mathbf{K}(\beta\bar{X}\beta\bar{Y})\bar{Z}, \quad P(\bar{X}, \bar{Y})\bar{Z} = \mathbf{K}(\beta\bar{X}, \gamma\bar{Y})\bar{Z}, \quad S(\bar{X}, \bar{Y})\bar{Z} = \mathbf{K}(\gamma\bar{X}, \gamma\bar{Y})\bar{Z}.$$

The contracted curvature tensors, denoted by \hat{R} , \hat{P} and \hat{S} respectively, are also known as the (v)h-, (v)hv- and (v)v-torsion tensors and are defined by

$$\hat{R}(\bar{X}, \bar{Y}) = R(\bar{X}, \bar{Y})\bar{\eta}, \quad \hat{P}(\bar{X}, \bar{Y}) = P(\bar{X}, \bar{Y})\bar{\eta}, \quad \hat{S}(\bar{X}, \bar{Y}) = S(\bar{X}, \bar{Y})\bar{\eta}.$$

Now, let (M, L) be a Finsler manifold, where L is the Lagrangian defining the Finsler structure on M . Let g be the Finsler metric in $\pi^{-1}(TM)$ defined by L .

Theorem 1.1. [8] *Let (M, L) be a Finsler manifold. There exists a unique regular connection ∇ in $\pi^{-1}(TM)$ such that*

- (a) ∇ is metric: $\nabla g = 0$,
- (b) The horizontal torsion of ∇ vanishes: $Q = 0$,
- (c) The mixed torsion T of ∇ satisfies $g(T(\bar{X}, \bar{Y}), \bar{Z}) = g(T(\bar{X}, \bar{Z}), \bar{Y})$.

Such a connection is called the Cartan connection associated to the Finsler manifold (M, L) .

One can show that the torsion of the Cartan connection has the property that $T(\bar{X}, \bar{\eta}) = 0$ for all $\bar{X} \in \mathfrak{X}(\pi(M))$. For the Cartan connection, we have

$$K\circ\gamma = id_{\pi^{-1}(TM)}, \quad \gamma\circ K = \begin{cases} id_{V(TM)} & \text{on } V(TM) \\ 0 & \text{on } H(TM) \end{cases} \quad (1.2)$$

Then, from (1.1) and (1.2), we get

$$\beta o\rho + \gamma oK = id_{T(TM)} \quad (1.3)$$

Hence, if we set $h := \beta o\rho$ and $v := \gamma oK$, then every vector field $X \in \mathfrak{X}(TM)$ can be represented uniquely in the form

$$X = hX + vX = \beta\rho X + \gamma KX \quad (1.4)$$

The maps h and v are the horizontal and vertical projectors associated with the Cartan connection ∇ : $h^2 = h$, $v^2 = v$, $h + v = id_{\mathfrak{X}(TM)}$, $voh = hov = 0$.

Definition 1.2. [10] *With respect to the Cartan connection ∇ , we have:*

– *The horizontal Ricci tensor Ric^h is defined by*

$$Ric^h(\overline{X}, \overline{Y}) := Tr\{\overline{Z} \mapsto R(\overline{X}, \overline{Z})\overline{Y}\}, \quad \text{for all } \overline{X}, \overline{Y} \in \mathfrak{X}(TM).$$

– *The horizontal Ricci map Ric_0^h is defined by*

$$g(Ric_0^h(\overline{X}), \overline{Y}) := Ric^h(\overline{X}, \overline{Y}), \quad \text{for all } \overline{X}, \overline{Y} \in \mathfrak{X}(TM).$$

– *The horizontal scalar curvature Sc^h is defined by*

$$Sc^h := Tr(Ric_0^h).$$

We terminate this section by some concepts and results concerning the notion of a nonlinear connection in the sense of Klein-Grifone [3], [4]:

Definition 1.3. *A nonlinear connection on M is a vector 1-form Γ on TM , C^∞ on TM , C^0 on TM , such that*

$$J\Gamma = J, \quad \Gamma J = -J.$$

The horizontal and vertical projectors h and v associated with Γ are defined by $h := \frac{1}{2}(I + \Gamma)$, $v := \frac{1}{2}(I - \Gamma)$. Thus Γ gives rise to the decomposition $TTM = H(TM) \oplus V(TM)$, where $H(TM) := Im h = Ker v$, $V(TM) := Im v = Ker h$. The torsion T of a nonlinear connection Γ is the vector 2-form on TM defined by $T := \frac{1}{2}[J, \Gamma]$. The curvature of a nonlinear connection Γ is the vector 2-form \mathfrak{R} on TM defined by $\mathfrak{R} := -\frac{1}{2}[h, h]$.

Proposition 1.4. *Let (M, L) be a Finsler manifold. The vector field G on TM determined by $i_G \Omega = -dE$ is a spray, called the canonical spray associated with the energy E , where $E := \frac{1}{2}L^2$ and $\Omega := dd_J E$.*

Theorem 1.5. *On a Finsler manifold (M, L) , there exists a unique conservative nonlinear connection ($d_h E = 0$) with zero torsion. It is given by:*

$$\Gamma = [J, G],$$

where G is the canonical spray.

Such a connection is called the Barthel connection or the Cartan nonlinear connection associated with (M, L) .

It should be noted that the horizontal and vertical projectors of the Cartan connection and the Barthel connection coincide. Also, the canonical spray $G = \beta o\overline{\eta}$, and G is thus horizontal.

2. π -Exterior derivative and \bar{d} -closed vector field

In this section, we introduce the notion of π -exterior differentiation operator \bar{d} , which is the Finslerian version of the (usual) exterior differentiation operator d of Riemannian geometry. We then investigate the \bar{d} -closed π -vector fields in Finsler geometry.

Let (M, L) be a Finsler manifold. Let g be the Finsler metric defined by the Lagrangian L and ∇ be the Cartan connection associated with (M, L) .

We start with the following lemma which is useful for subsequent use.

Lemma 2.1. *For all $\bar{X}, \bar{Y} \in \mathfrak{X}(\pi(M))$ and $X, Y \in \mathfrak{X}(TM)$, we have:*

- (a) $\widehat{R}(\bar{X}, \bar{Y}) = K[\beta\bar{X}, \beta\bar{Y}]$,
- (b) $\mathfrak{R}(X, Y) = -\gamma\widehat{R}(\rho X, \rho Y)$,

\mathfrak{R} being the curvature tensor of the Barthel connection.

Proof.

(a) One can easily show that

$$[\beta\bar{X}, \beta\bar{Y}] = \gamma(R(\bar{X}, \bar{Y})\bar{\eta}) + \beta(\nabla_{\beta\bar{X}}\bar{Y} - \nabla_{\beta\bar{Y}}\bar{X}). \quad (2.1)$$

Then, (a) follows directly from (2.1) by applying the operator K on both sides, noting that $K \circ \gamma = id_{\pi^{-1}(TM)}$ and $K \circ \beta = 0$.

(b) Setting $\bar{X} = \rho X$ and $\bar{Y} = \rho Y$ in (a), we get

$$\gamma\widehat{R}(\rho X, \rho Y) = \gamma K[\beta\rho X, \beta\rho Y] = v[hX, hY].$$

Then, (b) follows directly from the identity [13]: $\mathfrak{R}(X, Y) = -v[hX, hY]$. \square

Definition 2.2. *For any given π -vector field \bar{X} , the $\mathfrak{F}(TM)$ -linear operator $A_{\bar{X}}$ on $\mathfrak{X}(\pi(M))$ is defined by:*

$$A_{\bar{X}}(\bar{Y}) := \nabla_{\beta\bar{Y}}\bar{X}. \quad (2.2)$$

Definition 2.3. [9] *Let ω be a π -form of order $p \geq 0$.*

For $p > 0$, we define

$$\begin{aligned} (\bar{d}\omega)(\bar{X}_1, \dots, \bar{X}_{p+1}) &:= \sum_{i=1}^{p+1} (-1)^{i+1} \beta\bar{X}_i \cdot \omega(\bar{X}_1, \dots, \widehat{\bar{X}_i}, \dots, \bar{X}_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega(\rho[\beta\bar{X}_i, \beta\bar{X}_j], \bar{X}_1, \dots, \widehat{\bar{X}_i}, \dots, \widehat{\bar{X}_j}, \dots, \bar{X}_{p+1}) \end{aligned} \quad (2.3)$$

For $p = 0$, we set

$$(\bar{d}\omega)(\bar{X}) := \beta\bar{X} \cdot \omega, \text{ that is, } \bar{d}\omega = d\omega \circ \beta. \quad (2.4)$$

(Here, the symbol “ $\widehat{}$ ” means that the corresponding argument is omitted.)

In particular, for a (1) π -form ω , we have

$$(\bar{d}\omega)(\bar{X}, \bar{Y}) = \beta\bar{X} \cdot \omega(\bar{Y}) - \beta\bar{Y} \cdot \omega(\bar{X}) - \omega(\rho[\beta\bar{X}, \beta\bar{Y}]). \quad (2.5)$$

The operator \bar{d} will be called the π -exterior derivative.

Definition 2.4. *A $(p)\pi$ -form ω is said to be:*

- \bar{d} -closed if $\bar{d}\omega = 0$.
- \bar{d} -exact if $\omega = \bar{d}\alpha$, for some $(p-1)\pi$ -form α .

It should be noted that a \bar{d} -exact π -form is not necessarily \bar{d} -closed, contrary to the case of the (ordinary) exterior derivative d ([5] and [7]). This is due to the fact that the property $\bar{d}^2 = 0$ does not hold for the π -exterior derivative \bar{d} .

Definition 2.5. Let $\bar{X} \in \mathfrak{X}(\pi(M))$ be a π -vector field and let $\omega \in \mathfrak{X}^*(\pi(M))$ be the associated (1) π -form under the duality defined by the Finsler metric g : $\omega = i_{\bar{X}}g$. The π -vector field \bar{X} is said to be \bar{d} -closed (resp. \bar{d} -exact) if its associated π -form ω is \bar{d} -closed (resp. \bar{d} -exact).

The following result gives a simple characterization of \bar{d} -closed π -vector fields.

Theorem 2.6. A π -vector field \bar{X} is \bar{d} -closed if and only if the operator $A_{\bar{X}}$ is self-adjoint.

Proof. Let ω be the π -form associated to the π -vector field \bar{X} under the duality defined by the Finsler metric g . By (2.5), we have

$$\begin{aligned} (\bar{d}\omega)(\bar{Y}, \bar{Z}) &= \beta\bar{Y} \cdot \omega(\bar{Z}) - \beta\bar{Z} \cdot \omega(\bar{Y}) - \omega(\rho[\beta\bar{Y}, \beta\bar{Z}]) \\ &= \beta\bar{Y} \cdot g(\bar{X}, \bar{Z}) - \beta\bar{Z} \cdot g(\bar{X}, \bar{Y}) - g(\bar{X}, \rho[\beta\bar{Y}, \beta\bar{Z}]) \\ &= g(A_{\bar{X}}\bar{Y}, \bar{Z}) + g(\bar{X}, \nabla_{\beta\bar{Y}}\bar{Z}) - g(A_{\bar{X}}\bar{Z}, \bar{Y}) - g(\bar{X}, \nabla_{\beta\bar{Z}}\bar{Y}) \\ &\quad - g(\bar{X}, \rho[\beta\bar{Y}, \beta\bar{Z}]) \\ &= g(A_{\bar{X}}\bar{Y}, \bar{Z}) - g(A_{\bar{X}}\bar{Z}, \bar{Y}) + g(\bar{X}, Q(\bar{Y}, \bar{Z})). \end{aligned}$$

As the horizontal torsion tensor Q of the Cartan connection ∇ vanishes, the result follows. \square

Definition 2.7. The gradient of a function $f \in \mathfrak{F}(TM)$ is the π -vector field \bar{X} defined by:

$$i_{\bar{X}}g = \bar{d}f = df \circ \beta. \quad (2.6)$$

The gradient of the function f is denoted by $\text{grad } f$.

A π -vector field \bar{X} is said to be a gradient π -vector field if it is the gradient of some function $f \in \mathfrak{F}(TM)$: $\bar{X} = \text{grad } f$.

In Riemannian geometry, it is well known that ([5], [7], [12]) the gradient of any function $f \in \mathfrak{F}(M)$ is a closed vector field or, equivalently, A_X is self-adjoint. This result is not in general true in Finsler geometry. This is again due to the fact that $\bar{d}^2 \neq 0$. Nevertheless, we have

Theorem 2.8. The following assertions are equivalent:

- (a) The gradient of any function $f \in \mathfrak{F}(TM)$ is \bar{d} -closed, or, equivalently, gradient π -vector fields are \bar{d} -closed.
- (b) The curvature tensor \mathfrak{R} of the Barthel connection vanishes.
- (c) The horizontal distribution is completely integrable.
- (d) The π -exterior derivative \bar{d} has the property that $\bar{d}^2 = 0$.

Proof.

(a) \iff (b) : Let $f \in \mathfrak{F}(TM)$ be any arbitrary function and let $\overline{X} := \text{grad } f$. For all $\overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$ and for all $\overline{X} = \text{grad } f$, we have

$$\begin{aligned}
g(A_{\overline{X}}\overline{Y}, \overline{Z}) - g(A_{\overline{X}}\overline{Z}, \overline{Y}) &= g(\nabla_{\beta\overline{Y}}\overline{X}, \overline{Z}) - g(\nabla_{\beta\overline{Z}}\overline{X}, \overline{Y}) \\
&= \beta\overline{Y} \cdot g(\overline{X}, \overline{Z}) - g(\overline{X}, \nabla_{\beta\overline{Y}}\overline{Z}) \\
&\quad - \beta\overline{Z} \cdot g(\overline{X}, \overline{Y}) + g(\overline{X}, \nabla_{\beta\overline{Z}}\overline{Y}) \\
&= \beta\overline{Y} \cdot ((df \circ \beta)\overline{Z}) - \beta\overline{Z} \cdot ((df \circ \beta)\overline{Y}) \\
&\quad - (df \circ \beta)(\nabla_{\beta\overline{Y}}\overline{Z} - \nabla_{\beta\overline{Z}}\overline{Y}) \\
&= \beta\overline{Y} \cdot (\beta\overline{Z} \cdot f) - \beta\overline{Z} \cdot (\beta\overline{Y} \cdot f) - \beta(\nabla_{\beta\overline{Y}}\overline{Z} - \nabla_{\beta\overline{Z}}\overline{Y}) \cdot f \\
&= ([\beta\overline{Y}, \beta\overline{Z}] - \beta(\nabla_{\beta\overline{Y}}\overline{Z} - \nabla_{\beta\overline{Z}}\overline{Y})) \cdot f
\end{aligned}$$

In view of (2.1), the last equation takes the form

$$g(A_{\overline{X}}\overline{Y}, \overline{Z}) - g(A_{\overline{X}}\overline{Z}, \overline{Y}) = \gamma(R(\overline{Y}, \overline{Z})\overline{\eta}) \cdot f = \gamma(\widehat{R}(\overline{Y}, \overline{Z})) \cdot f \quad \forall f \in \mathfrak{F}(TM) \quad (2.7)$$

Now, the required equivalence follows from (2.7), taking into account Theorem 2.6, Lemma 2.1 and the fact that γ is a monomorphism.

(b) \iff (c) : This equivalence follows immediately from the identity [13]:

$$\mathfrak{R}(X, Y) = -v[hX, hY].$$

(c) \iff (d) : For all $f \in \mathfrak{F}(TM)$ and $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$, we have

$$(\overline{d}^2 f)(\overline{X}, \overline{Y}) = \beta\overline{X} \cdot (\beta\overline{Y} \cdot f) - \beta\overline{Y} \cdot (\beta\overline{X} \cdot f) - \beta(\rho[\beta\overline{X}, \beta\overline{Y}]) \cdot f = (I - \beta \circ \rho) \cdot [\beta\overline{X}, \beta\overline{Y}] \cdot f.$$

Hence, we have $(\overline{d}^2 f)(\overline{X}, \overline{Y}) = v[\beta\overline{X}, \beta\overline{Y}] \cdot f \quad \forall f \in \mathfrak{F}(TM)$.

The above equation, shows that the horizontal distribution is completely integrable if and only if $\overline{d}^2 = 0$ on $\mathfrak{F}(TM)$. This proves the implication (d) \implies (c). For the proof of the converse implication, refer to [9]. \square

Definition 2.9. A Finsler manifold (M, L) is said to be of scalar curvature κ if the $(v)h$ -torsion tensor \widehat{R} is written in the form:

$$\widehat{R} := R \otimes \overline{\eta} = \omega \wedge \phi, \quad (2.8)$$

where $\phi = I - L^{-1}\ell \otimes \overline{\eta}$; $\ell := L^{-1}i_{\overline{\eta}}g = dL \circ \gamma$ and $\omega := \frac{1}{3}L(L\overline{d}_{J\kappa} + 3\kappa\ell)$; $\overline{d}_{J\kappa}$ is the π -lift of $d_{J\kappa} = d\kappa \circ J$. (κ being the horizontal scalar curvature).

The formula (2.8) is found in [6] in a local coordinate form. It expresses the characteristic property of a Finsler space of scalar curvature.

Proposition 2.10. Let (M, L) be a Finsler manifold of scalar curvature κ . The gradient of any function $f \in \mathfrak{F}(TM)$ is \overline{d} -closed (equivalently, all gradient π -vector fields are \overline{d} -closed) if the horizontal scalar curvature κ vanishes.

Proof. The result follows from (2.7) and (2.8), taking Theorem 2.6 into account. \square

Theorem 2.11. Let (M, L) be a Finsler manifold of nonzero scalar curvature κ . Let $f \in \mathfrak{F}(TM)$ be an everywhere-nonzero positively homogeneous function of degree r in the directional arguments. If $\text{grad } f$ is \overline{d} -closed, then $f(x, y) = h(x)L^r$, for some function $h \in \mathfrak{F}(M)$.

Proof. Let $\overline{X} := \text{grad } f$. As \overline{X} is \overline{d} -closed, then by Theorem 2.6 and (2.7), we have

$$\gamma(R(\overline{Y}, \overline{Z})\overline{\eta}) \cdot f = 0.$$

But since (M, L) is of scalar curvature, then

$$\gamma((\omega \wedge \phi)(\overline{Y}, \overline{Z})) \cdot f = 0.$$

Setting $\overline{Z} = \overline{\eta}$ in the above equation, noting that $\phi(\overline{\eta}) = 0$, we get

$$\omega(\overline{\eta})(\gamma \phi(\overline{Y})) \cdot f = 0.$$

As $\omega(\overline{\eta}) = \frac{1}{3}L^2(\mathcal{C} \cdot \kappa + 3\kappa) \neq 0$, it follows that

$$(\gamma \phi(\overline{Y})) \cdot f = 0.$$

Consequently, by the definition of ϕ , we get

$$\gamma \overline{Y} \cdot f - L^{-1}\ell(\overline{Y})\mathcal{C} \cdot f = 0, \quad (2.9)$$

from which, since f is homogeneous of degree r ,

$$df \circ \gamma - r f L^{-1}\ell = 0, \text{ or } \frac{df}{f} \circ \gamma - r \frac{dL}{L} \circ \gamma = 0.$$

This equation is equivalent to

$$d(\log(fL^{-r})) \circ \gamma = 0.$$

Hence, fL^{-r} is independent of the directional arguments and, consequently, there exists a function $h \in \mathfrak{F}(M)$ so that $f = h(x)L^r$. \square

Corollary 2.12. *Under the hypothesis of Theorem 2.11, if the function $f \in \mathfrak{F}(TM)$ is homogeneous of degree zero in the directional arguments, then f is a function of positional arguments only, that is $f \in \mathfrak{F}(M)$.*

In fact, this follows from (2.9), since in this case $\mathcal{C} \cdot f = 0$.

Let U be an open subset of TM . An assignment

$$\mathfrak{D} : u \in U \longrightarrow \mathfrak{D}_u \subset P^{-1}(u) = \{u\} \times T_{\pi(u)}M,$$

such that every \mathfrak{D}_u is an m -dimensional vector subspace of $P^{-1}(u)$, is called an m -dimensional π -distribution on U . If \overline{X} is a π -vector field on U with $\overline{X}(u) \in \mathfrak{D}_u$ for every $u \in U$, we say that \overline{X} belongs to \mathfrak{D} and we write $\overline{X} \in \mathfrak{D}$.

For a given regular connection on $\pi^{-1}(TM)$, an m -dimensional π -distribution \mathfrak{D} on U is said to be h -involutive if for every π -vector fields \overline{X} and \overline{Y} , $\rho[\beta\overline{X}, \beta\overline{Y}]$ belongs to \mathfrak{D} whenever \overline{X} and \overline{Y} belong to \mathfrak{D} .

Definition 2.13. *Let \overline{X} be a given π -vector field on an open subset U of TM . The $(n-1)$ -dimensional π -distribution*

$$\mathfrak{D} : u \in U \longmapsto \mathfrak{D}_u := \{\overline{Y} \in P^{-1}(u) : g(\overline{X}, \overline{Y}) = 0\}$$

is called the π -distribution generated by (or associated with) \overline{X} .

Theorem 2.14. *If $\overline{X} \in \mathfrak{X}(\pi(M))$ is \overline{d} -closed, then the π -distribution \mathfrak{D} generated by \overline{X} is h -involutive.*

Proof. Suppose that \bar{Y} and \bar{Z} belong to \mathfrak{D} . As the h -torsion tensor of the Cartan connection ∇ vanishes, then

$$\begin{aligned} g(\rho[\beta\bar{Y}, \beta\bar{Z}], \bar{X}) &= g(\nabla_{\beta\bar{Y}}\bar{Z} - \nabla_{\beta\bar{Z}}\bar{Y}, \bar{X}) \\ &= \beta\bar{Y} \cdot g(\bar{Z}, \bar{X}) - g(\bar{Z}, \nabla_{\beta\bar{Y}}\bar{X}) - \beta\bar{Z} \cdot g(\bar{Y}, \bar{X}) + g(\bar{Y}, \nabla_{\beta\bar{Z}}\bar{X}). \end{aligned}$$

Since $g(\bar{X}, \bar{Y}) = 0 = g(\bar{X}, \bar{Z})$, it follows that

$$g(\rho[\beta\bar{Y}, \beta\bar{Z}], \bar{X}) = g(A_{\bar{X}}\bar{Z}, \bar{Y}) - g(\bar{Z}, A_{\bar{X}}\bar{Y}).$$

Then, by Theorem 2.6, \mathfrak{D} is h -involutive. \square

The following result characterizes certain \bar{d} -closed π -vector fields.

Proposition 2.15. *Let \bar{X} be a π vector field such that $i_{\bar{X}}\bar{d}g = 0$. Then, \bar{X} is \bar{d} -closed if and only if $\beta\bar{X}$ is an isometry.*

The result follows directly from the identity $\mathfrak{L}_{\beta\bar{X}}g = i_{\bar{X}}\bar{d}g + \bar{d}i_{\bar{X}}g$, where \mathfrak{L}_X is the Lie derivative with respect to $X \in \mathfrak{X}(TM)$ [9].

A generalized Randers manifold [11] is a Finsler manifold (M, L^*) whose Finsler structure L^* is given by $L^* = L + \alpha$, where L is a Finsler structure on M and $\alpha = g(\bar{b}, \bar{\eta})$; \bar{b} being the π -vector field defined in terms of a given 1-form δ on M by

$$\delta(X) = g(\bar{b}, \bar{X}) \quad \forall X \in \mathfrak{X}(M),$$

where \bar{X} is the π -life of X .

The change $L \longrightarrow L^* = L + \alpha$ is called a generalized Randers change.

The next result gives a characterization of \bar{d} -closeness of a remarkable π -vector field \bar{m} associated with Randers changes.

Proposition 2.16. *Let $L^* = L + \alpha$ be a generalized Randers change with closed α . The π -vector field $\bar{m} := \bar{b} - L^{-2}\alpha\bar{\eta}$ is \bar{d} -closed in the Finsler manifold (M, L^*) if and only if $\tau\bar{m}$ is \bar{d} -closed in the Finsler manifold (M, L) , where $\tau = L^*L^{-1}$.*

Proof. The relation between the Finsler metrics g and g^* associated with the Finsler structures L and L^* is given by [11]:

$$g^* = \tau(g - \ell \otimes \ell) + \ell^* \otimes \ell^*,$$

where $\tau = L^*L^{-1}$, $\ell = dL \circ \gamma$ and $\ell^* = \ell + d\alpha \circ \gamma$.

Now, if $d\alpha = 0$, then $\ell^* = \ell$ and consequently

$$g^* = \tau g + (1 - \tau)\ell \otimes \ell.$$

Then

$$i_{\bar{m}}g^* = \tau i_{\bar{m}}g + (1 - \tau)\ell(\bar{m})\ell.$$

As $\ell(\bar{m}) = 0$, as one can easily show, then the above relation reduces to $i_{\bar{m}}g^* = i_{\tau\bar{m}}g$, from which the result follows. \square

Two Finsler structure L and \tilde{L} on a manifold M are said to be conformal [14] if $\tilde{L} = e^{\sigma(x)}L$, for some function $\sigma(x)$ on M . The transformation $L \longrightarrow \tilde{L} = e^{\sigma(x)}L$ is called a conformal transformation (or a conformal change). The relation between the Finsler metrics g and \tilde{g} associated with L and \tilde{L} respectively is given by: $\tilde{g} = e^{2\sigma(x)}g$.

Let \bar{X} be a π -vector field, then $i_{\bar{X}}\tilde{g} = e^{2\sigma(x)}i_{\bar{X}}g$, and so

$$\bar{d}i_{\bar{X}}\tilde{g} = e^{2\sigma(x)}\bar{d}i_{\bar{X}}g + 2e^{2\sigma(x)}(d\sigma \otimes i_{\bar{X}}g = e^{2\sigma(x)}\bar{d}i_{\bar{X}}g + 2e^{2\sigma(x)}\frac{\partial\sigma}{\partial x^k}dx^k \otimes i_{\bar{X}}g.$$

Consequently, $\bar{d}i_{\bar{X}}\tilde{g} = e^{2\sigma(x)}\bar{d}i_{\bar{X}}g$ if and only if $\frac{\partial\sigma}{\partial x^k} = 0$, or equivalently, if and only if $\sigma(x)$ is a constant function (provided that M is connected). This proves the following result, which gives a characterization of homotheties in terms of \bar{d} -closed π -vector fields.

Theorem 2.17. *A \bar{d} -closed π -vector field remain \bar{d} -closed under a conformal transformation if and only if this transformation is a homothety.*

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